Announcements

1) Colin Adams student presentation 9:10
CB 2046, Wednesday

General audience talk: 3, Tuesday, CB 1030

Colloquium talk, 3, Wednesday, CB 2046

Notation: if

$$f : \mathbb{R}^n \to \mathbb{R}^m$$

is differentiable at $a \in \mathbb{R}^n$, set

$$\boxed{f'(a) = (Df)(a)}$$

Partial Derivatives

Let $f: \mathbb{R}^n \to \mathbb{R}^m$.

We can write

$$f = (f_1, f_2, \dots, f_m)$$

where $f_i : \mathbb{R}^n \to \mathbb{R}$

$\forall \quad 1 \leq i \leq m$.

You can think of

f_i as $f \cdot e_i$

where $\{e_i\}_{i=1}^{m}$ is the

standard basis for $\mathbb{R}^m$.

We define, for $1 \le i \le m$ and $1 \le j \le n$,

$$\boxed{\frac{\partial f_i}{\partial x_j}(a) = \lim_{t \to 0} \frac{f_i(a+te_j) - f_i(a)}{t}}$$

These are the <u>partial derivatives</u> of f_i, $1 \le i \le m$.

Theorem: (existence of partials)

Let $f: \mathbb{R}^n \to \mathbb{R}^m$ and suppose f is differentiable at $a \in \mathbb{R}^n$. Then if $f = (f_1, f_2, \ldots, f_m)$, $f_i : \mathbb{R}^n \to \mathbb{R}$

$\forall\ 1 \leq i \leq m$, $\frac{\partial f_i}{\partial x_j}(a)$ exists

$\forall\ 1 \leq i \leq m$, $1 \leq j \leq n$. Moreover,

$$\boxed{\frac{\partial f_i}{\partial x_j}(a) = \left(f'(a)\right)_{i,j}}$$

proof: Since $f'(a)$ exists,

we know

$$\lim_{h \to 0} \frac{\|f(a+h) - f(a) - f'(a)h\|_2}{\|h\|_2} = 0.$$

Hence,

$$\lim_{t \to 0} \frac{\|f(a+te_j) - f(a) - f'(a)(te_j)\|_2}{|t|} = 0 \quad \forall\ 1 \le j \le n$$

This means that, coordinate-wise,

$$\lim_{t \to 0} \frac{\left| f_i(a+te_j) - f_i(a) - (f'(a)te_j)_i \right|}{|t|} = 0$$

where $(f'(a)(te_j))_i$

is the i^{th} coordinate of the vector $f'(a)(te_j)$

This implies

$$\frac{\partial f_i}{\partial x_j}(a) \text{ exists}$$

and is equal to

$$(f'(a))_{i,j}.$$

$$(f'(a))_{i,j} = (f'(a)e_j) \cdot e_i \quad \square$$

Remark: The converse is not true in general and requires the additional assumption that

$$\frac{\partial f_i}{\partial x_j} \text{ are continuous}$$

in an open set containing the point $a \in \mathbb{R}^n$

$$\forall \ 1 \leq i \leq m, \ 1 \leq j \leq n.$$

<u>Recall:</u> (functions of one variable)

$f : \mathbb{R} \to \mathbb{R}$, suppose

f is differentiable

at $a \in \mathbb{R}$ and

$f'(a) > 0$. We

would like to conclude

that f is increasing

in an open interval about a.

But if f' is not continuous at $x=a$, we have some issues.

(e.g. $a=0$

$$f(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x=0 \end{cases}$$
)

We need to enforce some kind of continuity in our interval.

Suppose f' is continuous on $(a-c, a+c)$.

Then we can choose $0 < \delta < c$ so that if

$$|a-x| < \delta,$$

$$|f'(x) - f'(a)| < f'(a).$$

Then

$$-f'(a) < f'(x) - f'(a) < f'(a),$$

so adding $f'(a)$,

$$0 < f'(x) < 2f'(a).$$

Preliminaries to Inverse Function Theorem

Definition: (Convex set)

A subset $E \subseteq \mathbb{R}^n$

is convex if $\forall\, x, y \in E$,

$0 \le t \le 1$, $\boxed{tx + (1-t)y \in E}$.

Example 1: If we fix

$a \in \mathbb{R}^n$, $r > 0$,

$B(a, r)$ is convex.

Let $x, y \in B(a, r)$. Let

$0 \leq t \leq 1$, set

$z = tx + (1-t)y$.

Show $z \in B(a, r)$.

$$
\begin{aligned}
& d_2(z, a) \\
&= d_2\big(\overbrace{tx + (1-t)y}^{=z}, a\big) \\
&\leq d_2(tx, a) + d_2((1-t)y, a) \\
&= t\, d_2(x, a) + (1-t)\, d_2(y, a) \\
&< t r + (1-t) r = r.
\end{aligned}
$$

Cauchy-Schwarz Inequality

Let $x, y \in \mathbb{R}^n$.

Then

$$\boxed{x \cdot y \leq \|x\|_2 \cdot \|y\|_2}$$

Lemma: Let $E \subseteq \mathbb{R}^n$ be a convex, open set (in $\|\cdot\|_2$). Then if $f: \mathbb{R}^n \to \mathbb{R}^m$ is differentiable $\forall x \in E$ and satisfies $\sup_{x \in E} \|f'(x)\| \leq M$ for some $M \geq 0$, then $\forall x, y \in E$,

$$\boxed{\|f(x) - f(y)\|_2 \leq M \|x - y\|_2}$$

proof: Pick $a, b \in E$.

Define $\varphi : [0,1] \to E$

by $\varphi(t) = ta + (1-t)b$.

Since E is convex, $\varphi(t) \in E$

$\forall\ 0 \le t \le 1$.

Set $g(t) = f(\varphi(t))$.

By the chain rule,

$$g'(t) = f'(\varphi(t))\,\varphi'(t)$$

$\forall\ 0 \le t \le 1$.

Observe $\varphi'(t) = a-b$,

so

$$\|g'(t)\|_2 = \|f'(\varphi(t))(a-b)\|_2$$

$$\leq \|f'(\varphi(t))\| \, \|a-b\|_2$$

$$\leq M \, \|a-b\|_2 .$$

If we can prove that

$$\|g(1)-g(0)\|_2 \leq M\,\|a-b\|_2 ,$$

we will be done since

$\varphi(1) = a$, $\varphi(0) = b$

$$\Rightarrow \quad g(1) = f(\varphi(1))$$
$$= f(a)$$
$$g(0) = f(\varphi(0))$$
$$= f(b)$$

and so $\|f(a) - f(b)\|_2$
$$\leq M\|a-b\|_2$$

if we know $\|g(1) - g(0)\|_2$
$$\leq M\|a-b\|_2.$$